Final Exam — Group Theory (WIGT-07)

Tuesday January 24, 2016, 18.30h–21.30h

University of Groningen

Instructions

1. Write on each page you hand in your name and student number.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. Your grade for this exam is $E = (P + 10)/10$, where P is the number of points for this exam.

Problem 1 (20 points)

Set $V := \{(1), (12)(34), (13)(24), (14)(23)\} \subset S_4$.

a) Show that V is an abelian subgroup of S_4.

b) Show that V is a normal subgroup of S_4.

c) Show that S_4/V is isomorphic to S_3. *Hint:* The isomorphism theorem will **not** help in this case. Compute the sets σV for $\sigma \in S_4$ in order to construct an isomorphism $S_3 \to S_4/V$.

Problem 2 (15 points)

Let G be a cyclic group. Show that G is isomorphic to $\mathbb{Z}/m\mathbb{Z}$ for a uniquely determined $m \in \mathbb{Z}$ with $m \geq 0$.

Problem 3 (20 points)

Let G be a group and $H, N \subset G$ subgroups, where N is normal in G.

a) Show that $HN := \{hn | h \in H, n \in N\}$ is a subgroup of G.

b) In the case $G = \mathbb{Z}$, $H := 4\mathbb{Z}$ and $N := 6\mathbb{Z}$, find a subgroup $H' \subset H$ such that

$$H/H' \cong 2\mathbb{Z}/N.$$

Problem 4 (15 points)

Let G be a group with 148 elements. Show that G is not simple.

Problem 5 (20 points)

Let $H \subset \mathbb{Z}^3$ be the group generated by $(6, 6, 12), (0, 2, 4)$ and $(6, 10, 20)$.

a) Find a basis of H.

b) Find the rank and the elementary divisors of $\mathbb{Z}^3/H$.

End of test (90 points)

Solution to Problem 1 (20 points)

Set $V := \{(1), (12)(34), (13)(24), (14)(23)\} \subset S_4$.

a) Show that V is an abelian subgroup of S_4.
Solution: We see that $(1) \in V$, for any $\sigma \in V$ we have $\sigma^{-1} = \sigma \in V$ and by the fact $(12)(34)(13)(24) = (14)(23) = (13)(24)(12)(34)$ and the property that every element is of order 2 we get that V is closed under the operation. Since V is a group of order 4, it has to be abelian. (7pts)

b) Show that V is a normal subgroup of S_4.
Solution: V contains all products of two 2-cylcles of S_4. Since conjugation fixes the cycle-type, we get that V is fixed under conjugation. This is the same as saying V is normal. (7 pts)

c) Show that S_4/V is isomorphic to S_3. *Hint:* The isomorphism theorem will **not** help in this case. Compute the sets σV for $\sigma \in S_4$ in order to construct an isomorphism $S_3 \to S_4/V$.
Solution: Consider the map $\varphi : S_3 \to S4 \to S_4/V$ given by first using the standard injective homomorphism from S_3 to S_4, which sends an $\sigma \in S_3$ to the same permutation of four elements fixing the number 4. The second homomorphism is the canonical projection π. Since φ is the composition of two homomorphisms, it is itself a homomorphism. Further, φ is injective, since $\sigma \in \ker(\varphi)$ implies that $\sigma V = V$ and only $(1) \in V$ fixes 4. Thus, $\sigma = (1)$ which implies injectiveness of φ. By the fact that both groups consist of 6 elements, we also get φ is surjective; therefore, the statement holds.
Alternative: Construct the same homomorphism by computing the left cosets (or right cosets) of V. This shows that every class σV contains exactly one element fixing 4. This gives us the same homomorphism.
Alternative: Ignore the hint and try to apply the isomorphism theorem. For this, we need a surjective homomorphism $S_4 \to S_3$ with kernel equal to V. Looking into the appendix tells us that the rotations of a cube are isomorphic to S_4. Every rotation gives a permutation of the three axes of the cube. Any non-trivial rotation fixes the axes iff the rotation interchanges opposite sides of the cubes. Those are exactly the elements in V (under the isomorphism). Then the isomophism theorem yields the result. (6pts)

Solution to Problem 2 (15 points)

Let G be a cyclic group. Show that G is either isomorphic to $\mathbb{Z}/m\mathbb{Z}$ for a uniquely determined $m \in \mathbb{Z}$ with $m \geq 0$.
Solution: Since G is cyclic, we find $g \in G$ such that $\langle g \rangle = G$. (3pts) Thus, we have a surjective homomorphism $\Psi : \mathbb{Z} \to G$, where $\Psi(a) = x^a$. (3pts) By definition the kernel is generated by $\operatorname{ord}(x) = \operatorname{ord}(G)$ if $\operatorname{ord}(G) < \infty$ and trivial if $\operatorname{ord}(G) = \infty$.(6 pts) By the isomophism theorem we have $G \cong \mathbb{Z}/\ker(\Psi)$. (3pts)

Solution to Problem 3 (20 points)

Let G be a group and $H, N \subset G$ subgroups, where N is normal in G.

a) Show that $HN := \{hn|h \in H, n \in N\}$ is a subgroup of G.
Solution: We have $e \in HN$ since $e \in H$ and $e \in N$. For $hn \in HN$ we have $(hn)^{-1} = n^{-1}h^{-1} = h^{-1}hn^{-1}h^{-1} \in HN$. For $hn, h'n' \in HN$ we have $hnh'n' = h(h'h'^{-1})nh'n' \in HN$. (4 pts for each part)

b) In the case $G = \mathbb{Z}$, $H := 4\mathbb{Z}$ and $N := 6\mathbb{Z}$, find a subgroup $H' \subset H$ such that
$$H/H' \cong 2\mathbb{Z}/N.$$
Solution: We have $H + N = 2\mathbb{Z}$. By the isomophism theorem, we have $H/(H \cap N) \cong (H+N)/N$. By the lectures, we know $H \cap N = 12\mathbb{Z}$. Thus $H := 12\mathbb{Z}$ fulfills the requirements. (2pts per observation in total 8 pts)

Solution to Problem 4 (15 points)

Let G be a group with 148 elements. Show that G is not simple.
Solution: We have that $148 = 37 \cdot 4$. (2pts) Therefore, by the Sylow Theorem, there exist n_{37} Sylow 37-groups (2ts) such that $n_{37} \equiv 1 \pmod{37}$ and $n_{37}|4$. (2 pts) This implies $n_{37} = 1$ and there exists a unique Sylow 37-group. (3 pts) By the lectures, this group is normal and not equal to G or $\{0\}$. (3 pts) This implies G is not simple. (3 pts)

Solution to Problem 5 (20 points)

Let $H \subset \mathbb{Z}^3$ be the group generated by $(6, 6, 12), (0, 2, 4)$ and $(6, 10, 20)$.

a) Find a basis of H.

b) Find the rank and the elementary divisors of $\mathbb{Z}^3/H$.

Solution: We write the generators as columns of a matrix A and apply the algorithm that finds a basis for H and $\mathbb{Z}^3$ of a certain type. As a result we get the matrix
$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
(10 pts) Thus, the rank of H is equal to two. (2 pts) Since $(6, 10, 20) = (6, 6, 12) + 2(0, 2, 4)$ the first two vectors of the generationg set form a basis of H. (3pts) By the lectures we find that $\mathbb{Z}^3/H \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ (2 pts) and the searched rank is 1 and the elementary divisors are $(2, 6)$. (3 pts)